Depth with respect to a family of convex sets

L.I. Martínez Sandoval, R. Tamam

Ben-Gurion University of the Negev
Computational Geometry Seminar at TAU

Wednesday, March 15, 2017

Introduction

Depth

Question

Let $\mathcal{F}$ be a family of geometric objects and p a point. How "deep" is p with respect to $\mathcal{F}$?

Depth

Question

Let $\mathcal{F}$ be a family of geometric objects and p a point. How "deep" is p with respect to $\mathcal{F}$?

Applications in:

- Communications
- Statistics (detecting outliers)
- Motion planning
- Helly and Eckhoff type theorems

Example

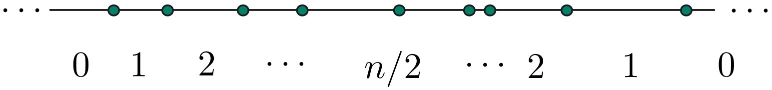


Figure: What is depth in $\mathbb{R}$?

Example

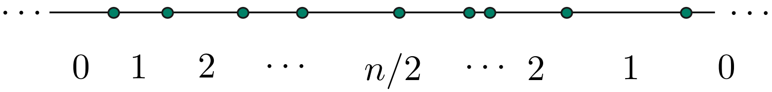


Figure: What is depth in $\mathbb{R}$?

- Points near the middle have greater depth, and depth decreases to 0 as we get far away.

Example

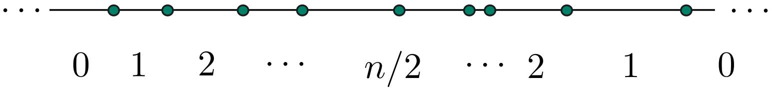


Figure: What is depth in $\mathbb{R}$?

- Points near the middle have greater depth, and depth decreases to 0 as we get far away.
- There is always a point with depth at least $\frac{n}{2}$, the median.

Tukey's depth

Definition (Tukey's depth, 1974)

Given a finite set of points S in $\mathbb{R}^d$, the Tukey depth of a point p with respect to S is defined as the minimum value of $|S \cap H|$ as H varies over the halfspaces H that contain p. We denote this value by $tdep_S(p)$.

Example Tukey's depth

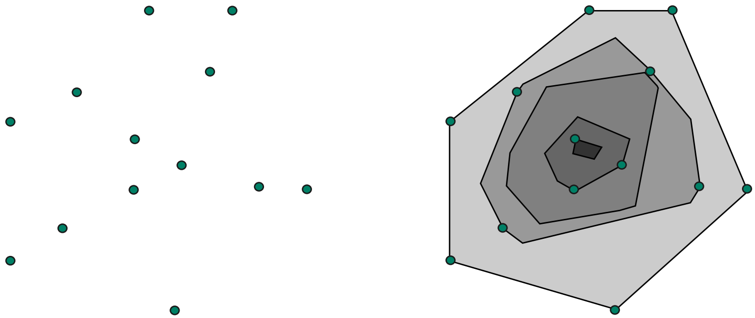


Figure: Example of Tukey's depth on the plane for 13 points

Depth with respect to a family of convex sets

Definition (-, RT, 2016)

Given a family of convex bodies $\mathcal{F}$ in $\mathbb{R}^d$, the family depth of a point p with respect to $\mathcal{F}$ is defined as the minimum value of

$$I_H := |\{F : F \in \mathcal{F},\ F \cap H \neq \emptyset\}|$$

as H varies over the halfspaces H that contain p. We denote this value by $dep_{\mathcal{F}}(p)$.

Example of family depth

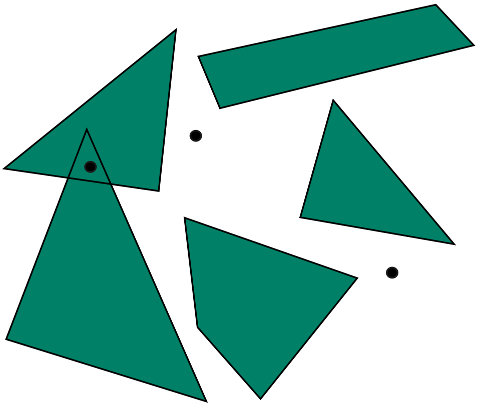


Figure: Example of family depth on the plane. What is the depth?

Example of family depth

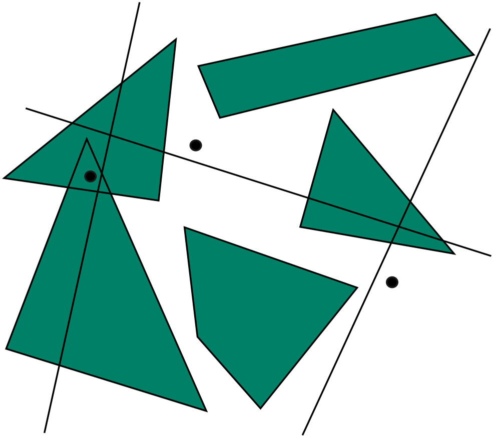


Figure: Depth is 2, 3, 1 from left to right

Basic properties & center regions

Good depth function

What should a "good depth function" satisfy?

Good depth function

What should a "good depth function" satisfy?
R. Liu (1990) and Y. Zuo, R. Serfling (2000):

- Affine invariance
- Maximality at the center For symmetric distributions
- Monotonicity Depth decreases away from a deepest point
- Vanishing at infinity

Good depth function

What should a "good depth function" satisfy?
R. Liu (1990) and Y. Zuo, R. Serfling (2000):

- Affine invariance
- Maximality at the center For symmetric distributions
- Monotonicity Depth decreases away from a deepest point
- Vanishing at infinity

Theorem (-,RT, 2016)

The function $dep_{\mathcal{F}}$ satisfies these four properties

Affine invariance

A depth function is affine invariant if for any affine transformation T we have $\text{dep}_{\mathcal{F}}(p) = \text{dep}_{T\mathcal{F}}(Tp)$.

Affine invariance

A depth function is affine invariant if for any affine transformation T we have $\mathrm{dep}_{\mathcal{F}}(p) = \mathrm{dep}_{T\mathcal{F}}(Tp)$.

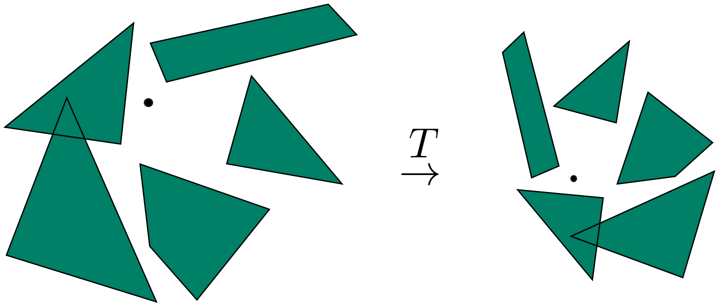


Figure: $\mathrm{dep}_{\mathcal{F}}$ is invariant under an affine transformation T

Maximality at center for symmetric families

A family $\mathcal{F}$ is symmetric with center p if for every F in $\mathcal{F}$, the symmetric set with respect to p is also in $\mathcal{F}$.

Maximality at center for symmetric families

A family $\mathcal{F}$ is symmetric with center p if for every F in $\mathcal{F}$, the symmetric set with respect to p is also in $\mathcal{F}$.

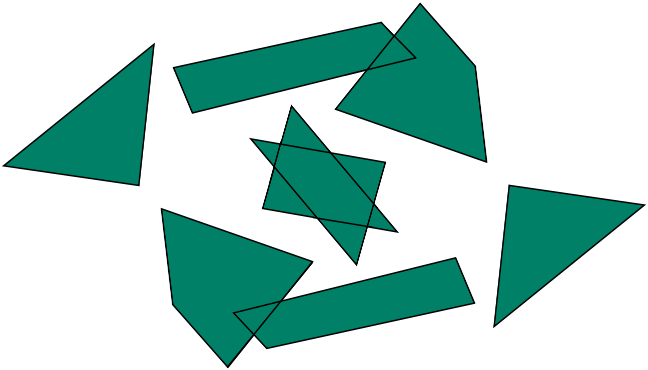


Figure: Maximality at the center

Maximality at center for symmetric families

A family $\mathcal{F}$ is symmetric with center p if for every F in $\mathcal{F}$, the symmetric set with respect to p is also in $\mathcal{F}$.

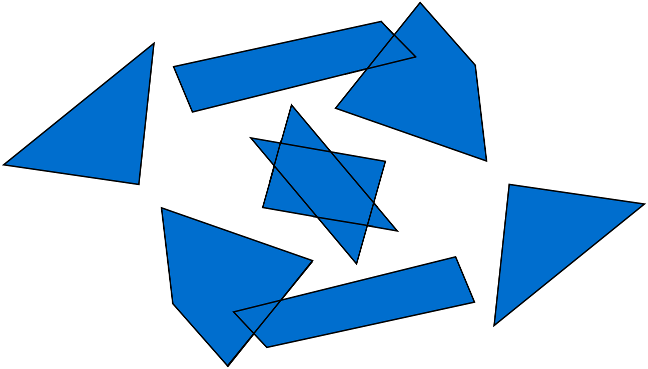


Figure: Maximality at the center

Monotonicity from the maximum

A depth function is monotonic from the maximum if $\text{dep}_{\mathcal{F}}(x)$ decreases as x gets away from a point y that maximizes $\text{dep}_{\mathcal{F}}$.

Monotonicity from the maximum

A depth function is monotonic from the maximum if $\mathrm{dep}_{\mathcal{F}}(x)$ decreases as x gets away from a point y that maximizes $\mathrm{dep}_{\mathcal{F}}$.

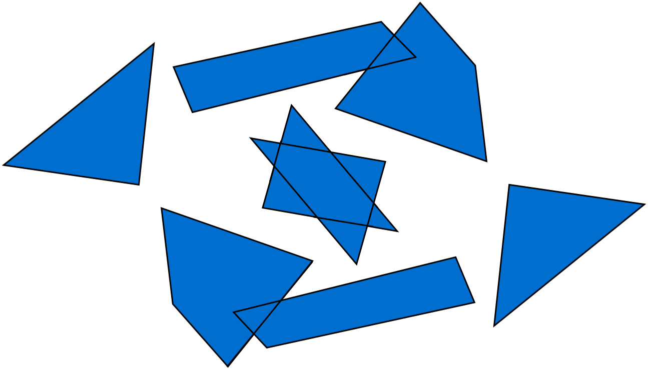


Figure: Monotonicity from the maximum

Vanishing at infinity

A depth function is vanishing at infinity if $\lim_{||x|| \to \infty} \mathrm{dep}_{\mathcal{F}}(x) = 0$.

Vanishing at infinity

A depth function is vanishing at infinity if $\lim_{||x|| \to \infty} \text{dep}_{\mathcal{F}}(x) = 0$.

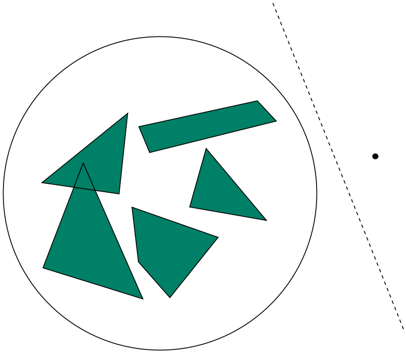


Figure: $\text{dep}_{\mathcal{F}}$ is vanishing at infinity

Connection to Tukey's depth

Both tdep and dep are good depth functions. How are they related?

Connection to Tukey's depth

Both tdep and dep are good depth functions. How are they related?

Proposition

- *Let S be a set of points in $\mathbb{R}^d$ and $\mathcal{S}$ be the family of singletons given by S. Then for any point p we have*

$$dep_{\mathcal{S}}(p) = tdep_S(p).$$

- *Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of convex sets in $\mathbb{R}^d$. Then for any point p we have*

$$dep_{\mathcal{F}}(p) \geq \sup_{S \colon \forall i\ |S \cap F_i| = 1} tdep_S(p).$$

- *In some cases the inequality above is strict*

Example

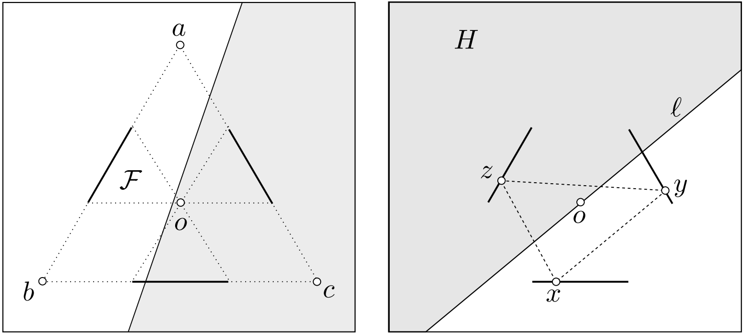


Figure: Example that shows that $\mathrm{dep}_{\mathcal{F}}(p)$ does not only depend on the Tukey depth of p with respect to representative sets of the family $\mathcal{F}$

Center regions

The *r-center* $C_r(\mathcal{F})$ is the set of points of depth at least r with respect to $\mathcal{F}$.

Center regions

The *r*-center $C_r(\mathcal{F})$ is the set of points of depth at least r with respect to $\mathcal{F}$.

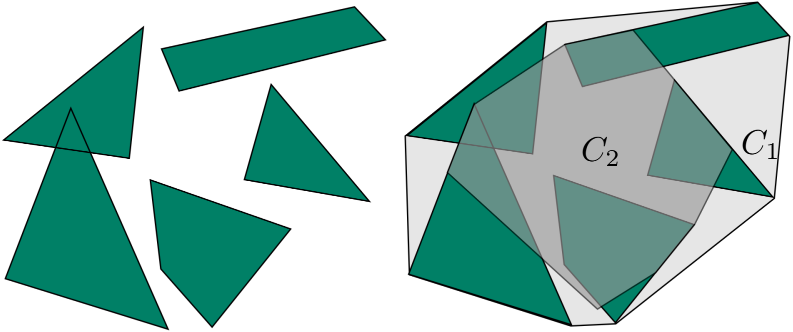


Figure: Example of center regions for family depth on the plane

Nice properties of center regions

The corresponding definition for Tukey's depth has been widely studied and is well behaved. For example:

1. The r-center is always convex.
2. The r-center is a polytope.
3. Each facet of the r-center lies in a hyperplane spanned by at most d points of S.
4. The halfspace opposite to the r-center with respect to one of such hyperplanes contains exactly $r - 1$ points.

Nice properties of center regions

The corresponding definition for Tukey's depth has been widely studied and is well behaved. For example:

1. The r-center is always convex.
2. The r-center is a polytope.
3. Each facet of the r-center lies in a hyperplane spanned by at most d points of S.
4. The halfspace opposite to the r-center with respect to one of such hyperplanes contains exactly $r-1$ points.

The r-center for $\mathrm{dep}_{\mathcal{F}}$ might fail to be a polytope, but it is always convex.

Nice properties of center regions

The corresponding definition for Tukey's depth has been widely studied and is well behaved. For example:

1. The r-center is always convex.
2. The r-center is a polytope.
3. Each facet of the r-center lies in a hyperplane spanned by at most d points of S.
4. The halfspace opposite to the r-center with respect to one of such hyperplanes contains exactly $r-1$ points.

The r-center for $\mathrm{dep}_{\mathcal{F}}$ might fail to be a polytope, but it is always convex. It is an intersection of planks.

Rotating planks

Lemma (The planks lemma)

Let n be a positive integer and r a real number in the interval $[0, n]$. Let $\mathcal{F}$ be a family of n convex sets of $\mathbb{R}^d$. Then for each direction u there exists a closed plank P_u perpendicular to it such that:

- *For each hyperplane in the plank, each of the halfspaces it defines intersects at least r of the sets of $\mathcal{F}$.*
- *For each of the two bounding hyperplanes of the plank, the halfspace defined by it that contains the plank,* **contains more** *than $n - r$ sets of $\mathcal{F}$.*

Furthermore, $C_r(\mathcal{F}) = \bigcap_{u \in \mathbb{S}^{d-1}} P_u$.

Proof of planks lemma

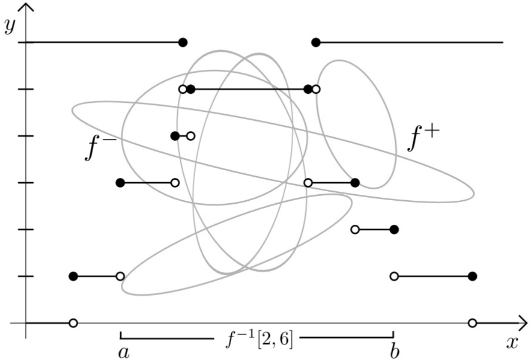


Figure: Example of projection of 6 convex sets to the x-axis. The image shows the functions f^- and f^+ for depth 2 overlapped on the family $\mathcal{F}$.

Centerpoint Theorem & Helly's Theorem

Centerpoint theorem

Theorem (Rado's centerpoint theorem, 1946)

For any finite set of points S in $\mathbb{R}^d$ we can always find a point of Tukey depth at least $\frac{|S|}{d+1}$.

Centerpoint theorem

Theorem (Rado's centerpoint theorem, 1946)

For any finite set of points S in $\mathbb{R}^d$ we can always find a point of Tukey depth at least $\frac{|S|}{d+1}$.

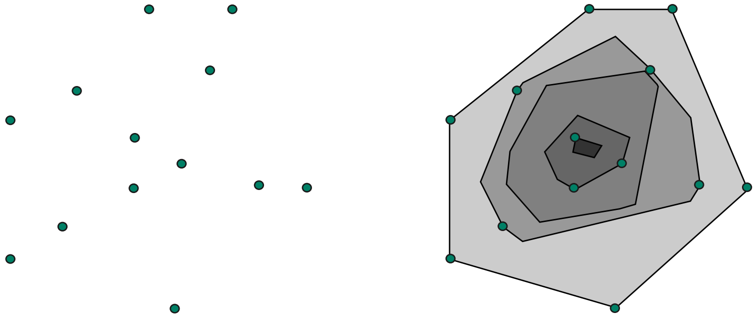


Figure: Centerpoint theorem for a set of 13 points

Intersection pattern

- Do we have a centerpoint theorem for $\mathrm{dep}_{\mathcal{F}}$?

Intersection pattern

- Do we have a centerpoint theorem for $\mathrm{dep}_{\mathcal{F}}$? Yes, at least we have CPT.

Intersection pattern

- Do we have a centerpoint theorem for $\mathrm{dep}_{\mathcal{F}}$? Yes, at least we have CPT.
- Can we do better?

Intersection pattern

- Do we have a centerpoint theorem for $\mathrm{dep}_{\mathcal{F}}$? Yes, at least we have CPT.
- Can we do better? Yes, by looking at the intersection pattern of $\mathcal{F}$.

Intersection pattern

- Do we have a centerpoint theorem for $\mathrm{dep}_{\mathcal{F}}$? Yes, at least we have CPT.
- Can we do better? Yes, by looking at the intersection pattern of $\mathcal{F}$.

Let's consider an extreme case.

Helly's theorem

Theorem (Helly's theorem, 1923)

Let $\mathcal{F}$ be a family of convex bodies in $\mathbb{R}^d$. If any subfamily of $\mathcal{F}$ with at most $d+1$ sets is intersecting, then the whole family $\mathcal{F}$ is intersecting.

Helly's theorem

Theorem (Helly's theorem, 1923)

Let $\mathcal{F}$ be a family of convex bodies in $\mathbb{R}^d$. If any subfamily of $\mathcal{F}$ with at most $d+1$ sets is intersecting, then the whole family $\mathcal{F}$ is intersecting.

Or, in other words, if a family of convex sets is non-intersecting, then we can find a subfamily of size at most $d+1$ that is non-intersecting.

Helly's theorem

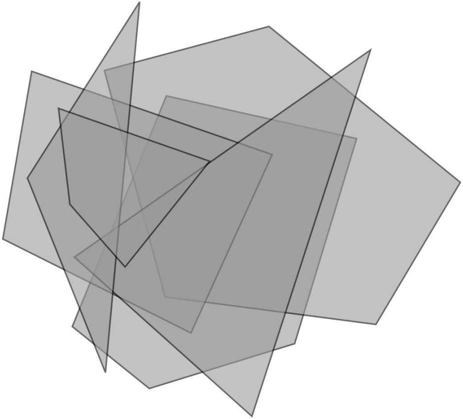


Figure: A family of non-intersecting convex sets

Helly's theorem

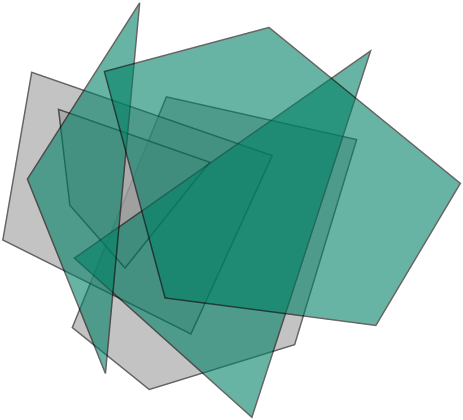


Figure: Three sets that detect that the family is non-intersecting

Main question

A family of convex bodies $\mathcal{F}$ is said to be *k-intersecting* if every subfamily of size k or less has non-empty intersection.

Main question

A family of convex bodies $\mathcal{F}$ is said to be k-intersecting if every subfamily of size k or less has non-empty intersection.

Problem

What is the largest value $\alpha_{d,k}$ so that any k-intersecting family convex bodies in $\mathbb{R}^d$ has a point of depth at least $\alpha_{d,k}|\mathcal{F}|$?

Main question

A family of convex bodies $\mathcal{F}$ is said to be k-intersecting if every subfamily of size k or less has non-empty intersection.

Problem

What is the largest value $\alpha_{d,k}$ so that any k-intersecting family convex bodies in $\mathbb{R}^d$ has a point of depth at least $\alpha_{d,k}|\mathcal{F}|$?

- $\alpha_{d,0} = \frac{1}{d+1}$ by Rado's centerpoint theorem
- $\alpha_{d,d+1} = 1$ by Helly's theorem
- In between, $\alpha_{d,k}$ interpolates between these two theorems

Naive approach

We can take one point from each of the $m = \binom{n}{k}$ intersections. We get a set S of m points. We apply the centerpoint theorem to S.

Naive approach

We can take one point from each of the $m = \binom{n}{k}$ intersections. We get a set S of m points. We apply the centerpoint theorem to S.

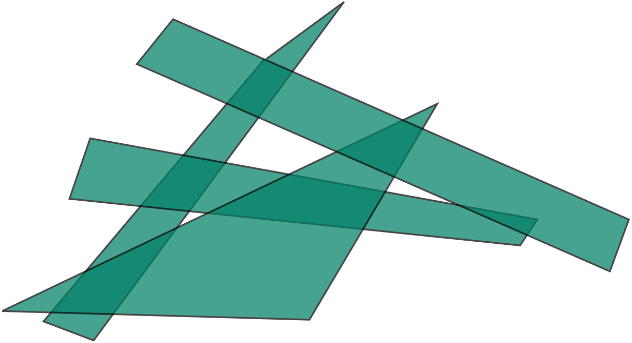


Figure: Pairwise intersecting sets

Naive approach

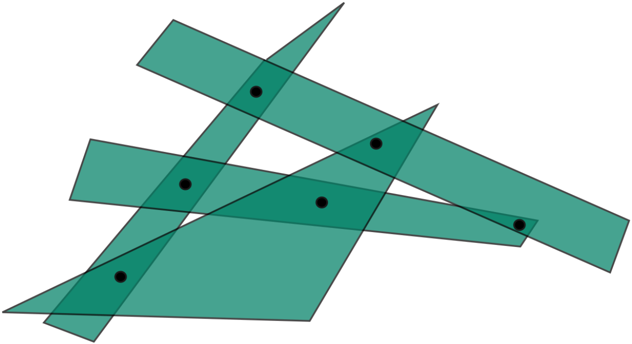


Figure: We take a point in each of the pairwise intersections to create S

Naive approach

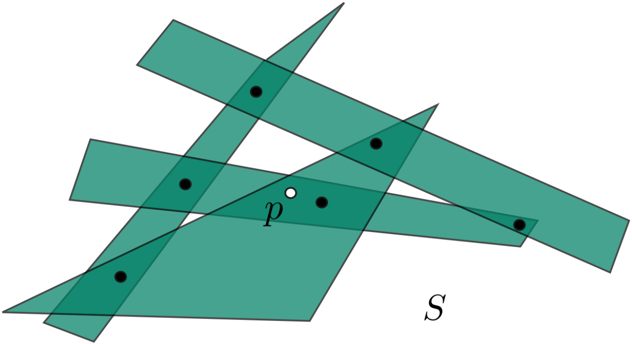


Figure: We consider a centerpoint p of S

Naive approach

Let H be a halfspace that contains p and intersects the minimum number of sets from $\mathcal{F}$, say j of them.

$$\frac{j^k}{k!} \approx \binom{j}{k} \geq |H \cap S| \geq \frac{|S|}{d+1} = \frac{1}{d+1}\binom{n}{k} \approx \frac{1}{d+1} \cdot \frac{n^k}{k!}$$

Naive approach

Let H be a halfspace that contains p and intersects the minimum number of sets from $\mathcal{F}$, say j of them.

$$\frac{j^k}{k!} \approx \binom{j}{k} \geq |H \cap S| \geq \frac{|S|}{d+1} = \frac{1}{d+1}\binom{n}{k} \approx \frac{1}{d+1} \cdot \frac{n^k}{k!}$$

So

$$\frac{j}{n} \gtrapprox \frac{1}{\sqrt[k]{d+1}}$$

And thus $\alpha_{d,k} \gtrapprox \frac{1}{\sqrt[k]{d+1}}$.

Naive approach

Let H be a halfspace that contains p and intersects the minimum number of sets from $\mathcal{F}$, say j of them.

$$\frac{j^k}{k!} \approx \binom{j}{k} \geq |H \cap S| \geq \frac{|S|}{d+1} = \frac{1}{d+1}\binom{n}{k} \approx \frac{1}{d+1} \cdot \frac{n^k}{k!}$$

So

$$\frac{j}{n} \gtrapprox \frac{1}{\sqrt[k]{d+1}}$$

And thus $\alpha_{d,k} \gtrapprox \frac{1}{\sqrt[k]{d+1}}$. We can do better.

Getting better than naive

Proposition

Let $\mathcal{F}$ be a family of 2-intersecting sets on the plane. Then there is a point such that any halfspace that contains it intersects at least $\frac{2}{3}|\mathcal{F}|$ of the sets.

Getting better than naive

Proposition

Let $\mathcal{F}$ be a family of 2-intersecting sets on the plane. Then there is a point such that any halfspace that contains it intersects at least $\frac{2}{3}|\mathcal{F}|$ of the sets.

We use the planks lemma & Helly's theorem. Explain on whiteboard.

The planks lemma

Lemma

Let n be a positive integer and r a real number in the interval $[0, n]$. Let $\mathcal{F}$ be a family of n convex sets of $\mathbb{R}^d$. Then for each direction u there exists a closed plank P_u perpendicular to it such that:

- *For each hyperplane in the plank, each of the halfspaces it defines intersects at least r of the sets of $\mathcal{F}$.*
- *For each of the two bounding hyperplanes of the plank, the halfspace defined by it that contains the plank, contains more than $n - r$ sets of $\mathcal{F}$.*

Furthermore, $C_r(\mathcal{F}) = \bigcap_{u \in \mathbb{S}^{d-1}} P_u(\mathcal{F})$.

Specific case of planks lemma

Lemma

Let n be a positive integer. Let $\mathcal{F}$ be a family of n convex sets on the plane. Then for each direction u there exists a closed plank P_u perpendicular to it such that:

- *For each line in the plank, each of the halfplanes it defines intersects at least $\frac{2}{3}n$ of the sets of $\mathcal{F}$.*
- *For each of the two bounding lines of the plank, the line defined by it that contains the plank, contains more than $\frac{n}{3}$ sets of $\mathcal{F}$.*

Furthermore, $C_{\frac{2}{3}n}(\mathcal{F}) = \bigcap_{u \in \mathbb{S}^1} P_u(\mathcal{F})$.

The main theorem

A purely combinatorial hitting set problem

Definition

Let m be a positive integer and k an integer in $[m]$. We define $\beta_{m,k}$ as the smallest real number β for which the following holds. For any finite set X and any m of its subsets $A_1, \ldots, A_m$ with $|A_i| > \beta \cdot |X|$ $(i = 1, 2, \ldots, m)$ there exists a hitting set of size at most k.

A purely combinatorial hitting set problem

Definition

Let m be a positive integer and k an integer in $[m]$. We define $\beta_{m,k}$ as the smallest real number β for which the following holds. For any finite set X and any m of its subsets $A_1, \ldots, A_m$ with $|A_i| > \beta \cdot |X|$ $(i = 1, 2, \ldots, m)$ there exists a hitting set of size at most k.

Intuitively, if we fix m and k, then any collection of m sufficiently large subsets of any set X can be hit with k elements.

Understanding β_m, k

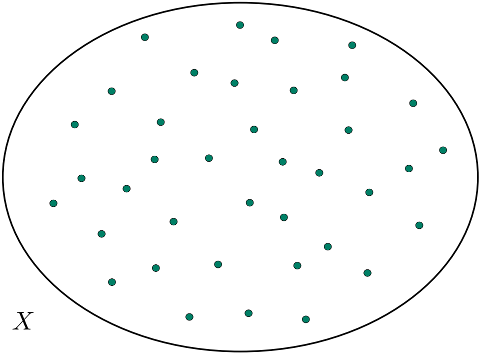


Figure: Any finite set X. Suppose that for any family of 4 subsets we can find a hitting set of size 2. How large they have to be to always be able to do this?

Understanding β_m, k

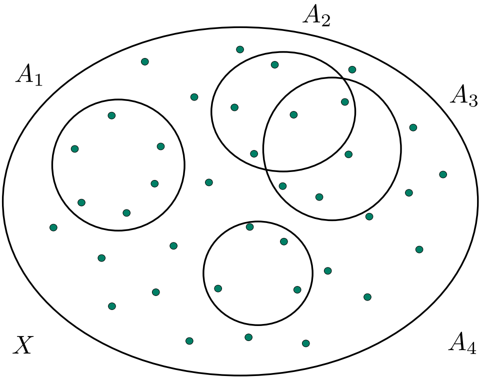


Figure: If they are proportionally small with respect to $|X|$, we can fit them and make them "very disjoint" and hard to hit.

Understanding β_m, k

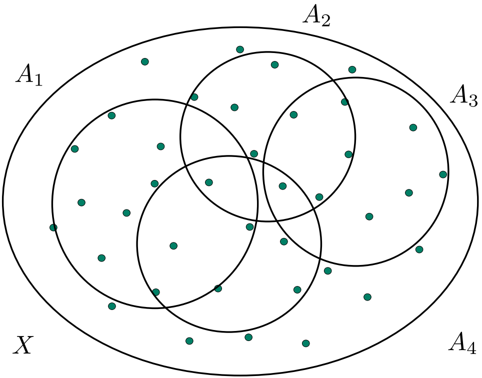


Figure: If they are proportionally large with respect to $|X|$, they start to overlap and they are easier to hit with few elements.

Main theorem: A curious connection

Theorem (-,RT, 2016)

For any positive integer d *and an integer* k *in* $[d+1]$ *we have:*

$$\alpha_{d,k} + \beta_{d+1,k} = 1.$$

Sketch of the proof

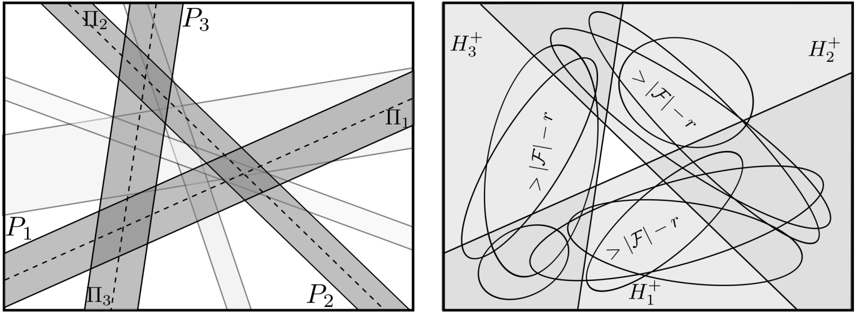


Figure: Sketch of the proof of Main Theorem

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1 - \beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

Sketch of the proof

Proof (One side, by contradiction).

$$\text{No depth } (1-\beta_{d+1,k})|\mathcal{F}| \text{ \& Planks lemma \& Helly's theorem}$$
$$\downarrow$$
$$\text{Halfspaces } H_1,\ldots,H_{d+1} \text{ containing more than } \beta_{d+1,k}|\mathcal{F}| \text{ each}$$

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1,\dots,H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1,\dots,A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1,\dots,H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1,\dots,A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

Hitting set of size k for $A_1,\dots,A_{d+1}$

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1, \ldots, H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1, \ldots, A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

Hitting set of size k for $A_1, \ldots, A_{d+1}$

$\downarrow$

k convex sets of the family $F_1, \ldots, F_k$ & k-intersection hypothesis

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1, \ldots, H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1, \ldots, A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

Hitting set of size k for $A_1, \ldots, A_{d+1}$

$\downarrow$

k convex sets of the family $F_1, \ldots, F_k$ & k-intersection hypothesis

$\downarrow$

Point p in $\cap F_i$

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1, \ldots, H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1, \ldots, A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

Hitting set of size k for $A_1, \ldots, A_{d+1}$

$\downarrow$

k convex sets of the family $F_1, \ldots, F_k$ & k-intersection hypothesis

$\downarrow$

Point p in $\cap F_i \subseteq \cap H_i$.

Sketch of the proof

Proof (One side, by contradiction).

No depth $(1-\beta_{d+1,k})|\mathcal{F}|$ & Planks lemma & Helly's theorem

$\downarrow$

Halfspaces $H_1, \ldots, H_{d+1}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

$A_1, \ldots, A_{d+1}$ subsets of $\mathcal{F}$ containing more than $\beta_{d+1,k}|\mathcal{F}|$ each

$\downarrow$

Hitting set of size k for $A_1, \ldots, A_{d+1}$

$\downarrow$

k convex sets of the family $F_1, \ldots, F_k$ & k-intersection hypothesis

$\downarrow$

Point p in $\cap F_i \subseteq \cap H_i$. Contradiction!

$\square$

Consequences

Corollary (CPT applied to a combnatorial hitting set)

The value of $\beta_{m,k}$ *is in* $1 - \Omega\left(\frac{1}{\sqrt[k]{m}}\right)$

Consequences

Corollary (CPT applied to a combnatorial hitting set)

The value of $\beta_{m,k}$ *is in* $1-\Omega\left(\frac{1}{\sqrt[k]{m}}\right)$

Corollary (Exact "almost Helly")

For any positive integer d $\beta_{d,d+1}=\frac{1}{d+1}$ *and thus* $\alpha_{d,d}=\frac{d}{d+1}$.

Consequences

Corollary (CPT applied to a combnatorial hitting set)

The value of $\beta_{m,k}$ *is in* $1 - \Omega\left(\frac{1}{\sqrt[k]{m}}\right)$

Corollary (Exact "almost Helly")

For any positive integer d $\beta_{d,d+1} = \frac{1}{d+1}$ *and thus* $\alpha_{d,d} = \frac{d}{d+1}$.

Proof.

Among $d+1$ subsets of $|X|$ with more than $\frac{1}{d+1}|X|$ elements each, two must intersect. Create the hitting set with a common point and one point from each of the remaining sets. $\square$

Consequences

Corollary (Exactly in between Helly and Rado)

For any k, $\beta_{2k,k} \leq 1 - \frac{1}{\sqrt[k]{15}}$ and thus $\alpha_{2k-1,k} \geq \frac{1}{\sqrt[k]{15}}$.

Consequences

Corollary (Exactly in between Helly and Rado)

For any k, $\beta_{2k,k} \leq 1 - \frac{1}{\sqrt[k]{15}}$ and thus $\alpha_{2k-1,k} \geq \frac{1}{\sqrt[k]{15}}$.

Proof (Probabilistic method with blemishes).

We need to hit $2k$ subsets of X using k elements.

Consequences

Corollary (Exactly in between Helly and Rado)

For any k, $\beta_{2k,k} \leq 1 - \frac{1}{\sqrt[k]{15}}$ and thus $\alpha_{2k-1,k} \geq \frac{1}{\sqrt[k]{15}}$.

Proof (Probabilistic method with blemishes).

We need to hit $2k$ subsets of X using k elements. We sample randomly $0.63k$ elements of X.

Consequences

Corollary (Exactly in between Helly and Rado)

For any k, $\beta_{2k,k} \leq 1 - \frac{1}{\sqrt[k]{15}}$ and thus $\alpha_{2k-1,k} \geq \frac{1}{\sqrt[k]{15}}$.

Proof (Probabilistic method with blemishes).

We need to hit $2k$ subsets of X using k elements. We sample randomly $0.63k$ elements of X.

$$\mathbb{P}(\text{Sample} \cap A_i \neq \emptyset) \geq 1 - \left(\frac{1}{\sqrt[k]{15}}\right)^{0.63k}$$

$$\mathbb{E}(\#\text{ of } A_i\text{'s hit}) \geq 2k\left(1 - \left(\frac{1}{\sqrt[k]{15}}\right)^{0.63k}\right) > 1.63k.$$

$\square$

Connections to transversal theorems

Transversals

A transversal line (plane, hyperplane, etc) for a family $\mathcal{F}$ of sets is a line that intersects each set of the family.

Transversals

A transversal line (plane, hyperplane, etc) for a family $\mathcal{F}$ of sets is a line that intersects each set of the family.

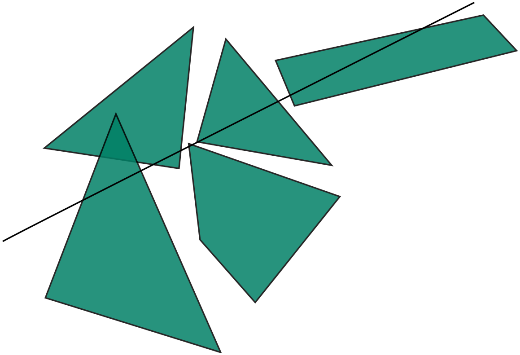


Figure: Transversal line

Looking for transversal lines

Question (Classic)

Is there a Helly-type theorem for transversal line on the plane? Is there value of k for which the following happens?

Looking for transversal lines

Question (Classic)

Is there a Helly-type theorem for transversal line on the plane? Is there value of k for which the following happens?

If every k sets of a family of convex sets have a transversal line, then the whole family has a transversal line.

Looking for transversal lines

Question (Classic)

Is there a Helly-type theorem for transversal line on the plane? Is there value of k for which the following happens?
If every k sets of a family of convex sets have a transversal line, then the whole family has a transversal line.

No.

Looking for transversal lines

Question (Classic)

Is there a Helly-type theorem for transversal line on the plane? Is there value of k for which the following happens?
If every k sets of a family of convex sets have a transversal line, then the whole family has a transversal line.

No. But there are trade-offs.

Transversal theorems

Theorem (Hadwiger, 1957 and subsequent work)

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. If we can "order" the sets of $\mathcal{F}$ and for each $d+1$ sets give a transversal consistent with the ordering, then $\mathcal{F}$ has a transversal hyperplane.

Transversal theorems

Theorem (Hadwiger, 1957 and subsequent work)

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. If we can "order" the sets of $\mathcal{F}$ and for each $d+1$ sets give a transversal consistent with the ordering, then $\mathcal{F}$ has a transversal hyperplane.

Theorem (Katchalski, Liu, 1980)

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^2$ and $k \geq 3$. If each k sets of $\mathcal{F}$ sets have a transversal line, then there is a line through at least $\gamma_k|\mathcal{F}|$ sets of $\mathcal{F}$.

A new transversal theorem

We change the line hypothesis for a k-intersection hypothesis.
This of course guarantees a transversal hyperplane.

A new transversal theorem

We change the line hypothesis for a k-intersection hypothesis. This of course guarantees a transversal hyperplane. But there is an interesting trade-off result.

Proposition

Let d be a positive integer and k an integer in $\{2, \ldots, d+1\}$. Let $\mathcal{F}$ be a k-intersecting finite family of convex sets in $\mathbb{R}^d$. Then there exists a point such that any hyperplane through it is transversal to at least $\alpha_{d,k}|\mathcal{F}|$ sets of $\mathcal{F}$.

Holmsen's tight triple theorem

A tight triple consists of three convex sets A, B, C for which

$$\mathrm{conv}(A \cup B) \cap \mathrm{conv}(B \cup C) \cap \mathrm{conv}(C \cup A) \neq \emptyset.$$

Remark: Three convex sets with transversal line are a tight triple.

Holmsen's tight triple theorem

A tight triple consists of three convex sets A, B, C for which

$$\mathrm{conv}(A \cup B) \cap \mathrm{conv}(B \cup C) \cap \mathrm{conv}(C \cup A) \neq \emptyset.$$

Remark: Three convex sets with transversal line are a tight triple.

Theorem (Holmsen, 2013)

Let $\mathcal{F}$ be a finite family of convex sets for which any of its triples is tight. Then there is a transversal line to at least $\frac{1}{8}|\mathcal{F}|$ sets of $\mathcal{F}$.

Using family depth to give a simpler proof

Proposition

If any triple of a family $\mathcal{F}$ of convex sets is tight, then there is a point with family depth at least $\frac{1}{2}|\mathcal{F}|$.

Using family depth to give a simpler proof

Proposition

If any triple of a family $\mathcal{F}$ of convex sets is tight, then there is a point with family depth at least $\frac{1}{2}|\mathcal{F}|$.

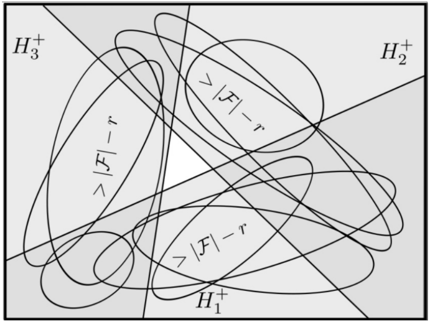


Figure: The plank lemma once more!

Alternative proof

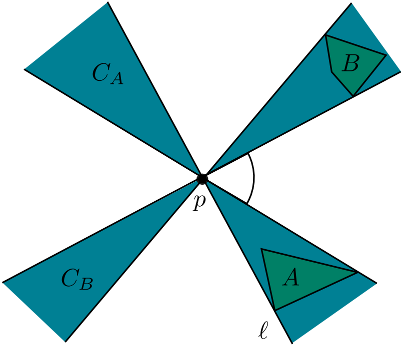
Proof.

Let p be a point of depth at least $\frac{1}{2}|\mathcal{F}|$. For each set A consider the minimal double cone C_A with apex p that contains it.

Alternative proof

Proof.

Let p be a point of depth at least $\frac{1}{2}|\mathcal{F}|$. For each set A consider the minimal double cone C_A with apex p that contains it.



□

Open problems

1. Study the algorithmic aspects of finding family depth centerpoints or centerpoint regions for, say, a family of n polygons with m edges in total.

Open problems

1. Study the algorithmic aspects of finding family depth centerpoints or centerpoint regions for, say, a family of n polygons with m edges in total.
2. Give a detailed probabilistic analysis that finds the correct asymptotic value of $\alpha_{d,k}$

Thank you!

Thank you for your attention!

Thank you!

Thank you for your attention!