The convex dimension of k-uniform hypergraphs

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Abstract

The convex dimension of a k-uniform hypergraph is the smallest dimension d for which there is an injective mapping of its vertices into $\mathbb{R}^d$ such that the set of k-barycenters of all hyperedges is in convex position.

We completely determine the convex dimension of complete k-uniform hypergraphs. This settles an open question by Halman, Onn and Rothblum, who solved the problem for complete graphs. We also provide lower and upper bounds for the number of hyperedges of k-uniform hypergraphs on n vertices with convex dimension d.

To prove these results we restate them in terms of affine projections that preserve the vertices of the hypersimplex, and generalize them to projections that preserve higher dimensional skeleta.

1 Introduction

Motivated by certain convex combinatorial optimization problems, Halman, Onn and Rothblum [6] defined a *convex embedding* of a k-uniform hypergraph $H = (V, E)$ into $\mathbb{R}^d$ as an injective map $f : V \to \mathbb{R}^d$ such that the set of *k-barycenters*

$$f(E) := \left\{ \frac{1}{k} \sum_{v \in e} f(v) : e \in E \right\}$$

is in convex position (i.e. each point is a vertex of the convex hull of $f(E)$); and the *convex dimension* $\mathsf{cd}(H)$ of H as the minimal d for which a convex embedding of H into $\mathbb{R}^d$ exists.

Their article focused on graphs, the $k = 2$ case. They studied the problem of determining the convex dimension for specific families of graphs: paths, cycles, complete graphs and bipartite graphs. And they also investigated the extremal problem of determining the maximum number of edges that a graph on n vertices and fixed convex dimension can have. The latter problem has been studied afterwards by several authors, in particular because of its strong relation with the problem of determining large convex subsets in Minkowski sums [2, 4], see [5] and references therein.

For values of $k > 2$, the only result of which we are aware of is the upper bound $\mathsf{cd}(H) \le 2k$ for any k-uniform hypergraph H, proved by Halman et al. by mapping the vertices onto points on the moment curve in $\mathbb{R}^{2k}$ [6]. The convex-hull of all k-barycenters has also been studied under the name of *k-set polytope* in relation to the study of k-sets and j-facets, see [1].

Our main result is the complete determination of the convex dimension of $K_n^{(k)}$, the complete k-uniform hypergraph on n vertices, for any k, $1 \le k \le n-1$.

Theorem 1 *Given positive integers n and k such that $2 \le k \le n-2$, we have that*

$$\mathsf{cd}(K_n^{(k)}) = \begin{cases} 2k & \text{if } n \ge 2k+2, \\ n-2 & \text{if } n \in \{2k-1, 2k, 2k+1\}, \\ 2n-2k & \text{if } n \le 2k-2. \end{cases}$$

Also, $\mathsf{cd}(K_2^{(1)}) = 1$ and $\mathsf{cd}(K_n^{(1)}) = \mathsf{cd}(K_n^{(n-1)}) = 2$ for $n \ge 3$.

This matches and extends the results in [6], where it is proved that $\mathsf{cd}(K_n) = 4$ for $n \ge 6$. Table 1 shows the explicit values of $\mathsf{cd}(K_n^{(k)})$ given by Theorem 1 for small values of n and k.

$k \setminus n$	2	3	4	5	6	7	8	9	10	11	12
1	1	2	2	2	2	2	2	2	2	2	2
2		2	2	3	4	4	4	4	4	4	4
3			2	3	4	5	6	6	6	6	6
4				2	4	5	6	7	8	8	8
5					2	4	6	7	8	9	10
6						2	4	6	8	9	10
7							2	4	6	8	10

Table 1: First values of $\mathsf{cd}(K_n^{(k)})$. Red values correspond to the cases $n \ge 2k+2$ or $n \le 2k-2$. White values correspond to the cases $n \in \{2k-1, 2k, 2k+1\}$, when $k \ge 2$. Green values correspond to the cases $k = 1$ and $k = n-1$.

We provide a polyhedral proof of Theorem 1. Namely, we reformulate the existence of a convex em-

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bedding of $K_n^{(k)}$ into $\mathbb{R}^d$ in terms of polytope projections that preserve the vertices of the hypersimplex $\Delta_{n,k}$, that is, the polytope whose vertices are the $\binom{n}{k}$ incidence vectors of k-subsets of $[n]$. Once the problem is restated in terms of polytope projections, we proceed as in the framework used by Sanyal when studying the number of vertices of Minkowski sums [8], based on the projection lemma [10].

Let $g_k(n,d)$ be the maximum number of hyperedges that a k-uniform hypergraph on n vertices of convex dimension d can have. We have $g_k(n,d) \leq \binom{n}{k}$, and that $g_k(n,d) = \binom{n}{k}$ for any $d \geq 2k$ by Theorem 1. By combining Theorem 1 and de Caen's bound on Turán numbers for hypergraphs [3] we get sharper upper bounds for g_k when $d \leq 2k-1$, as n grows.

Theorem 2 *For $1 \leq d \leq 2k-1$ we have*

$$g_k(n,d) \leq c_{d,k,n} \cdot n^k + o(n^k),$$

where $c_{d,k,n}$ are coefficients that satisfy:

$$\lim_{n\to\infty} (1 - k! \cdot c_{d,k,n})^{-1} = \begin{cases} \binom{d+2}{k-1} & \text{if } d \geq 2k-3 \\ \binom{\lfloor d/2 \rfloor + k}{k-1} & \text{if } d \leq 2k-4. \end{cases}$$

Finally, we give a lower bound for g_k through the following embeddability result.

Theorem 3 *There is a convex embedding of the complete k-uniform k-partite hypergraph $K_{n,n,\dots,n}^{(k)}$ into $\mathbb{R}^{k+1}$. Therefore, for fixed $d \geq k+1$ we have that $g_k(n,d)$ is in $\Theta(n^k)$ as $n \to \infty$.*

2 Projections that strictly preserve the vertices of the hypersimplex

In this section we reformulate Theorem 1 in terms of polytope projections that preserve vertices.

Definition 4 (Definition 3.1 in [10]) *Let P be a polytope and $\pi : P \to \pi(P)$ a linear projection. A face $F \subseteq P$ is strictly preserved under π if $\pi(F)$ is a face of $\pi(P)$ combinatorially isomorphic to F; and $\pi^{-1}(\pi(F)) = F$.*

For the restatement of Theorem 1 we use the following auxiliary lemma. Recall that the *(n,k)-hypersimplex* is the polytope:

$$\Delta_{n,k} = \operatorname{conv}\Big\{ x \in \{0,1\}^n \,\Big|\, \sum_{1 \leq i \leq n} x_i = k \Big\}.$$

Lemma 5 *The existence of a convex embedding of $K_n^{(k)}$ into $\mathbb{R}^d$ is equivalent to the existence of a linear projection of the hypersimplex $\Delta_{n,k}$ to $\mathbb{R}^d$ that strictly preserves its $\binom{n}{k}$ vertices.*

Proof. For $n \geq k \geq 1$, let $V = \{v_1, \dots, v_n\}$ be the vertex set of $K_n^{(k)}$. To any embedding $f : V \to \mathbb{R}^d$ we associate the linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ given by $\pi(k \cdot e_i) = f(v_i)$. Notice that π maps the vertices of $\Delta_{n,k}$ to the barycenters of k-subsets of $f(V)$. These are in convex position if and only if all the vertices of $\Delta_{n,k}$ are strictly preserved by π. $\square$

For a d-polytope $P \subset \mathbb{R}^d$ and a linear projection $\pi : \mathbb{R}^d \to \mathbb{R}^e$, the Projection Lemma [10, Prop. 3.2] gives a criterion to characterize which faces of P are strictly preserved by π in terms of the associated projection $\tau : \mathbb{R}^d \to \mathbb{R}^{d-e}$ onto the kernel of π (cf. [8, Sec. 3.2]).

Lemma 6 (Projection Lemma [10, Prop. 3.2]) *Let $P \subset \mathbb{R}^d$ be a d-polytope, $\pi : \mathbb{R}^d \to \mathbb{R}^e$ a linear projection, and $\tau : \mathbb{R}^d \to \mathbb{R}^{d-e}$ be the associated projection onto the kernel of π.*

Let $F \subset P$ be a face of P and let $\{\mathbf{n}_i \,|\, i \in I\}$ be the normal vectors to the facets of P that contain F. Then F is strictly preserved if and only if $\{\tau(\mathbf{n}_i) \,|\, i \in I\}$ positively span $\mathbb{R}^{d-e}$; i.e. if $0 \in \operatorname{int}\operatorname{conv}\{\tau(\mathbf{n}_i) \,|\, i \in I\}$.

One last ingredient is the dimension and hyperplane description of $\Delta_{n,k}$, which are well known (see for example [9, Ex. 0.11]).

Lemma 7 *The hypersimplex $\Delta_{n,k}$ is $(n-1)$-dimensional, has $\binom{n}{k}$ vertices, $2n$ facets and the inequality description*

$$\Delta_{n,k} = \Big\{ \sum_{i\in[n]} x_i = k \Big\} \cap \bigcap_{i\in[n]} \Big\{ x_i \geq 0 \Big\} \cap \bigcap_{i\in[n]} \Big\{ x_i \leq 1 \Big\}.$$

From here, we proceed as follows. Assume that there is a *good projection* $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^d$ that strictly preserves all the vertices of $\Delta_{n,k}$. Then Lemma 6 would ensure certain positive dependencies on the vector configuration induced by the image of the normal vectors to facets of $\Delta_{n,k}$ under the projection τ to the kernel of π. We state explicitly these dependencies below.

By the description in Lemma 7, $\Delta_{n,k}$ has $2n$ facets whose normal vectors we may pair up as $\{\mathbf{m}_i, \mathbf{n}_i\}$, where $\mathbf{m}_i$ corresponds to the inequality $x_i \geq 0$ and $\mathbf{n}_i$ corresponds to the inequality $x_i \leq 1$ for $i \in [n]$. They satisfy

$$\mathbf{m}_i + \mathbf{n}_i = 0 \text{ for } i \in [n] \text{ and } \sum_{i\in[n]} \mathbf{m}_i = \sum_{i\in[n]} \mathbf{n}_i = 0.$$

Combining Lemma 6 with the facial structure of the hypersimplex, we get:

Lemma 8 *The existence of a good projection π with associated normal projection τ implies that $\tau(\{\mathbf{n}_i, \mathbf{m}_j \,|\, i,j \in [n]\})$ is an $(n-d-1)$-dimensional configuration of vectors with the following strictly positive dependencies:*

a) $0 \in \operatorname{int}\operatorname{conv}\{\tau(\mathbf{m}_i) : i \in [n]\}$,

b) $0 \in \operatorname{int}\operatorname{conv}\{\tau(\mathbf{n}_i) : i \in [n]\}$,

c) $0 \in \operatorname{int}\operatorname{conv}\{\tau(\mathbf{m}_i), \tau(\mathbf{n}_i)\}$ *for* $i \in [n]$,

d) $0 \in \operatorname{int}\operatorname{conv}(\{\tau(\mathbf{m}_i) : i \in J\} \cup \{\tau(\mathbf{n}_i) : i \in I\})$ *for every disjoint* $I, J \subset [n]$, $|I| = k$, $|J| = n - k$.

e) $0 \in \operatorname{int}\operatorname{conv}(\{\tau(\mathbf{m}_i) : i \in I\} \cup \{\tau(\mathbf{n}_i) : i \in J\})$ *for every disjoint* $I, J \subset [n]$, $|I| = k$, $|J| = n - k$.

Note that the configuration of vectors is symmetric around the origin. This has another important interpretation that we will use later on.

Corollary 9 *A good projection* $\pi : \mathbb{R}^{n-1} \to \mathbb{R}^d$ *exists for* $\Delta_{n,k}$ *if and only if it exists for* $\Delta_{n,n-k}$.

Of course, $\Delta_{n,k}$ and $\Delta_{n,n-k}$ are affinely equivalent, so Corollary 9 should not be too unexpected. However, the fact that $\mathsf{cd}(K_n^{(k)}) = \mathsf{cd}(K_n^{(n-k)})$ is not entirely obvious from the definition of cd. It also has an alternative short geometric proof. Suppose f is a convex embedding of $K_n^{(k)}$ into $\mathbb{R}^d$. Consider the barycenter b of $f(V)$. The barycenter a of any k-subset of $f(V)$, the barycenter c of the complementary $(n-k)$-subset and b are collinear. The segment ac is split in ratio $k : n-k$ by b. Therefore, the set of $(n-k)$-barycenters is a homothetic copy of the set of k-barycenters. Since the second is in convex position, the first one is as well.

3 Proof of Theorem 1

By Corollary 9, we may assume from now on that $n \geq 2k$.

The case $k = 1$ is easy, as k-barycenters degenerate to the vertices of the set, and we need them to be in convex position. For $n = 2$ we need dimension 1, and for $n \geq 3$, dimension 2 is enough, as we may take any n-gon. So we may assume $k \geq 2$.

By the definition of cd, we have monotonicity on n, because if n increases, we are required to preserve more hyperedges. In other words,

$$\mathsf{cd}(K_{n'}^{(k)}) \geq \mathsf{cd}(K_n^{(k)}) \text{ for } n' \geq n.$$

Hence, it is enough to prove that:

$$\mathsf{cd}(K_n^{(k)}) = n - 2 \text{ for } n \in \{2k, 2k+1, 2k+2\}.$$

Indeed, this implies by monotonicity that

$$\mathsf{cd}(K_n^{(k)}) \geq 2k \text{ for } n \geq 2k+2.$$

This inequality is tight by [6]. We obtain a valid construction by mapping the vertices of Δ_{n-1} to points in the moment curve in $\mathbb{R}^{2k}$. We get a k-neighborly polytope, and thus the k-barycenters of the projected vertices are in convex position.

The lower bound for the key cases $n \in \{2k, 2k+1, 2k+2\}$ is given by the following lemma.

Lemma 10 *For* $k \geq 2$ *and* $l \in \{0, 1, 2\}$, *the hypersimplex* $\Delta_{2k+l,k}$ *has no codimension* 2 *projection that preserves all its vertices.*

Due to space constraints we provide only a sketch of the proof. By Lemma 8, a codimension 2 projection implies the existence of a specific configuration of vectors in $\mathbb{R}^2$ with specific strictly positive dependencies. Using a halving line, it is possible to show that all these dependencies cannot hold simultaneously.

To finish the proof, we need a construction that matches the lower bound.

Lemma 11 *Every hypersimplex* $\Delta_{n,k}$ *of dimension at least* 3 *has a codimension* 1 *projection that strictly preserves all its vertices.*

This is obvious for $k \in \{1, n-1\}$, and for $2 \leq k \leq n-2$, it can be verified by showing that the vertices of an $(n-2)$-dimensional simplex with an interior point form a set of n points all whose k-barycenters are in convex position.

4 Hypergraphs with many barycenters in convex position

Now we turn our attention to the maximum number of barycenters in convex position that a uniform hypergraph of fixed dimension may have. First, we give a lower bound by mapping a complete k-partite k-uniform hypergraph. For this we use the following particular version of a result by Matschke, Pfeifle and Pilaud:

Theorem 12 (Theorem 2.6 in [7]) *There are sets* $I_1, \ldots, I_k \subset \mathbb{R}$, *with* $|I_i| = n$ *for all* i *such that the polytope*

$$P := \operatorname{conv}\{(a_1, a_2, \ldots, a_k, \sum_{i \in [k]} a_i^2)\},$$

where $(a_1, \ldots, a_k)$ *ranges over* $I_1 \times \cdots \times I_k$, *has all its possible* n^k *vertices.*

Proof. [Proof of Theorem 3] Let

$$I_i = \{a_{i1}, \ldots, a_{in}\} \subset \mathbb{R} \text{ for } i \in [k]$$

be the sets given by Theorem 12. Let $V = V_1 \cup \cdots \cup V_k$ be the vertex set of $\Delta_{n,n,\ldots,n}^{(k)}$, where

$$V_i = \{v_{i1}, \ldots, v_{in}\} \text{ for } i \in [k].$$

Consider the mapping $f : V \to \mathbb{R}^{k+1}$ given by

$$f(v_{ij}) = k \cdot (a_{ij} \cdot \mathbf{e}_i + a_{ij}^2 \mathbf{e}_{k+1}).$$

Any hyperedge from $\Delta_{n,n,\ldots,n}^{(k)}$ is obtained by choosing one vertex from each V_i, so every k-barycenter is precisely of the form

$$(a_1, a_2, \ldots, a_k, \sum_{i \in [k]} a_i^2)$$

for $a_i \in I_i$. By Theorem 12, all these barycenters lie in convex position, so f is indeed a convex embedding into $\mathbb{R}^{k+1}$. □

Now we focus on upper bounds for g_k. Fix k and $1 \leq d \leq 2k-1$. Using Theorem 1, we obtain that the maximum value $n = n_{d,k}$ so that $K_n^{(k)}$ has a convex embedding into $\mathbb{R}^d$ is for $d \geq 2$

$$n_{d,k} = \begin{cases} d+2 & \text{if } d \in \{2k-3, 2k-2, 2k-1\}, \\ \lfloor \frac{d}{2} \rfloor + k & \text{if } 1 \leq d \leq 2k-4. \end{cases} \tag{1}$$

and for $d = 1$, $n_{1,1} = 1$, $n_{1,2} = 2$ and $n_{1,k} = k-1$ for $k \geq 3$.

The first values of $n_{d,k}$ are contained in Table 4.

$k \setminus d$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	4	5							
3	2	4	5	6	7					
4	3	5	5	6	7	8	9			
5	4	6	6	7	7	8	9	10	11	
6	5	7	7	8	8	9	9	10	11	12
7	6	8	8	9	9	10	10	11	11	12

Table 2: Values of $n_{d,k}$ for small values of d and k.

We recall the following bound for Turán numbers for complete hypergraphs by de Caen [3]:

Theorem 13 *A k-uniform hypergraph with no complete $K_\ell^{(k)}$ as an induced subhypergraph can have at most*

$$EX(n,k,\ell) = \frac{1}{k!}\left(1 - \frac{n-\ell+1}{n-k+1} \cdot \frac{1}{\binom{\ell-1}{k-1}}\right) n^k + o(n^k)$$

edges.

The proof of Theorem 2 follows from 13 and the values given in (1). Due to space constraints, we omit the details.

5 Concluding remarks

The results in this abstract form part of a larger project in which we study projections that preserve skeleta of hypersimplices. In terms of k-sets, these higher dimensional faces of projected hypersimplices correspond to (i,j)-partitions, cf. [1].

An extended analysis of the sketched technique yields the following more general result. Let $d = d(n,k,i)$ be the smallest dimension for which we can find a projection $\pi : \Delta_{n,k} \to \mathbb{R}^d$ that strictly preserves the i-skeleton of $\Delta_{n,k}$.

Theorem 14 *Given integers $n \geq k \geq 1$ and $0 \leq i \leq n-1$, the value of $d(n,k,i)$ is determined as follows. Let*

$$A_{n,i} = \{0,1,2,\ldots,i+1\} \cup \{n-i-1, n-i, \ldots, n\}$$
$$B_{k,i} = \{2k-2i-1, \ldots, 2k+2i+1\}.$$

Then

$$d(n,k,i) = \begin{cases} 2i+2k & \text{if } n \geq 2i+2k+2 \\ 2n-2i+2k & \text{if } n \leq 2i-2k-2 \\ n-1 & \text{if } n \in B_{i,k},\ k \in A_{n,i} \\ n-2 & \text{if } n \in B_{i,k},\ k \notin A_{n,i} \end{cases}$$

For values of $i \geq 1$ an extended version of Lemma 10 has to be used, and an additional behaviour appears for values of k smaller than $i+1$ and larger than $n-k-1$, similar to what happens in the results above for $k=1$ and $k=n-1$. Similarly, the constructions for upper bounds require a more careful analysis. The details are much more involved and out of the scope of this abstract.

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